Math 760

# Chapter 3 HW

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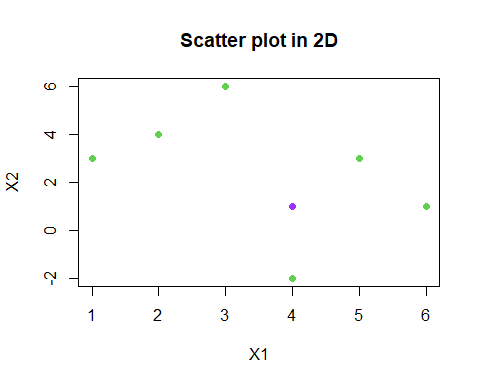
## 2. Given the data matrix

### (a) Graph the scatter plot in p = 2 dimensions, and locate the sample mean on your diagram.

The sample mean matrix is:

## [,1]  
## [1,] 4  
## [2,] 1

Now, we can graph the scatter plot, and note that the sample mean is colored in purple.



### (b) Sketch the n = 3-space representation of the data, and plot the deviation vectors and .

Let’s first find the deviation vectors and .

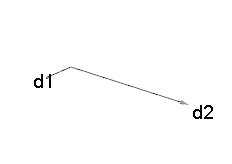
Our sample mean matrix is:

## [,1]  
## [1,] 4  
## [2,] 1

But because X is a 3x2 matrix, we will need to make the sample mean as such. Then, we can subtract the X and mean matrix to get:

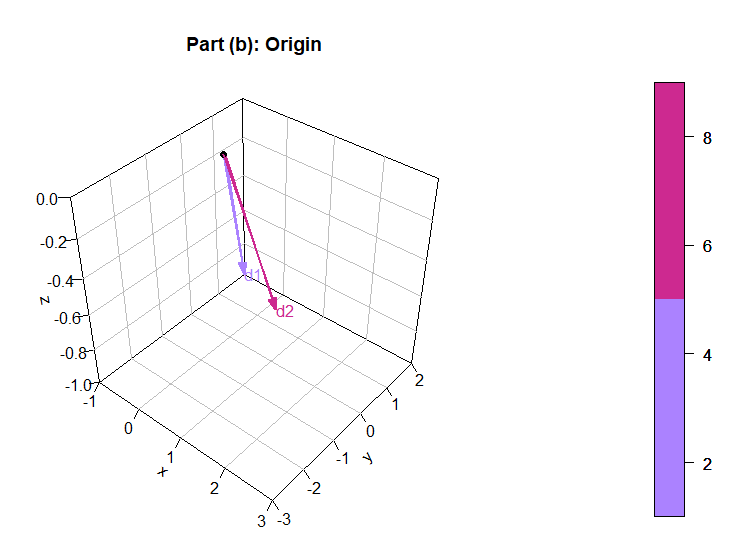
## d1 d2  
## [1,] -1 3  
## [2,] 2 -3  
## [3,] -1 0

Now, we can plot them in a 3D space.



### (c) Sketch the deviation vectors in (b) emanating from the origin. Calculate their lengths and the cosine of the angle between them. Relate these quantities to and R.

Let’s actually put the deviation vectors into a proper 3D graph.



With this in mind, let’s calculate their lengths and the cosine of the angle between them. From (3-5), we get the squared lengths, so we’ll square root them.

## The length of d1 is 2.44949

## The length of d2 is 4.242641

With this in mind, we can find the matrix. The formulas are:

## s11 is 2

## s22 is 6

## s12 is -3

## [,1] [,2]  
## [1,] 2 -3  
## [2,] -3 6

Let be the angle between and , then, cos() is:

## [1] -0.8660254

cos() is also known as the sample correlation coefficient; thus, **R** is:

## [,1] [,2]  
## [1,] 1.0000000 -0.8660254  
## [2,] -0.8660254 1.0000000

## 

## 6. Consider the data matrix

### 

### (a) Calculate the matrix deviations (residuals), . Is this matrix of full rank? Explain.

We will need to first find the , which is:

## [,1]  
## [1,] 2  
## [2,] 3  
## [3,] 1

Then, we can calculate the residuals, which are:

## [,1] [,2] [,3]  
## [1,] -3 0 -3  
## [2,] 0 1 1  
## [3,] 3 -1 2

The matrix deviations are as follows:

## d1 is [ -3 0 -3 ]

## d2 is [ 0 1 1 ]

## d3 is [ 3 -1 2 ]

And because they aren’t the same, the matrix is not of full rank.

### 

### (b) Determine S and calculate the generalized sample variance |S|. Interpret the latter geometrically.

To find **S**, we can multiply the residuals twice, then divide it by 2.

## [,1] [,2] [,3]  
## [1,] 9.0 -1.5 -7.5  
## [2,] -1.5 1.0 0.5  
## [3,] -7.5 0.5 7.0

Now we’ll calculate |**S**|, which is:

## [1] 5.995204e-15

This means that the 3 deviation vectors lie in a 2D subspace, and the 3D volume enclosed by the deviation vectors is 0.

### (c) Using the results in (b), calculate the total sample variance. [See (3-23)]

According to (3-23), the total sample variance = . So, the total sample variance is:

## [1] 17

## 

## 10. When the generalized variance is zero, it is the columns of the mean corrected data matrix that are linearly dependent, not necessarily those of the data matrix itself. Given the data

### 

### (a) Obtain the mean corrected data matrix, and verify that the columns are linearly dependent. Specify an vector that establishes the dependence.

The matrix is:

## [,1]  
## [1,] 5  
## [2,] 2  
## [3,] 3

Now, we can obtain the mean corrected data matrix, which is:

## [,1] [,2] [,3]  
## [1,] -2 -1 -3  
## [2,] 1 2 3  
## [3,] -1 0 -1  
## [4,] 2 -2 0  
## [5,] 0 1 1

And notice, if we added up the first 2 columns of the mean corrected data matrix, they equal the third column. This means, if we want , then matrix a will have to be . Therefore, the columns are linearly dependent.

## [,1] [,2] [,3]  
## [1,] 1 1 -1

## [,1] [,2] [,3] [,4] [,5]  
## [1,] -2 1 -1 2 0  
## [2,] -1 2 0 -2 1  
## [3,] -3 3 -1 0 1

## [,1] [,2] [,3] [,4] [,5]  
## [1,] 0 0 0 0 0

### 

### (b) Obtain the sample covariance S, and verify that the generalized variance is zero.

The sample covariance matrix is:

## [,1] [,2] [,3]  
## [1,] 2.5 0.0 2.5  
## [2,] 0.0 2.5 2.5  
## [3,] 2.5 2.5 5.0

And using vector a from (a), Sa = 0. Thus, the generalized variance is zero.

## [,1] [,2] [,3]  
## [1,] 0 0 0

### 

### (c) Show that the columns of the data matrix are linearly independent in this case.

Let’s start with this formula: Xa = 0.

We have our data matrix:

## [,1] [,2] [,3]  
## [1,] 3 1 0  
## [2,] 6 4 6  
## [3,] 4 2 2  
## [4,] 7 0 3  
## [5,] 5 3 4

And our A matrix:

And when multiplied, we have this:

With these equations, we can use them to show that all a’s equal 0.

We have proven , then plugging it into the first two equations:

Therefore, the columns of the data matrix are linearly independent.

## 

## 15. Repeat Exercise 3.14 using the data matrix

**and the linear combinations**

### (a) Evaluate the sample means, variances, and covariance b’X and c’X from first principles. That is, calculate the observed values of b’X and c’X, and then use the sample mean, variance, and covariance formulas.

From first principles, we will start calculating the observed values of **b’X**.

## The observed value of bX1 is: 8

## The observed value of bX2 is: 14

## The observed value of bX3 is: 14

Then, we can find the sample mean:

## The sample mean of bX is: 12

And finally, the sample variance:

## The sample variance of bX is: 12

Now, let’s do the observed values, sample mean, and sample variance of **c’X**.

## The observed value of cX1 is: 44

## The observed value of cX2 is: 68

## The observed value of cX3 is: 47

## The sample mean of cX is: 53

## The sample variance of cX is: 171

Finally, let’s find the sample covariance of **b’X** and **c’X**.

## The sample covariance of b'X and c'X is: 27

### 

### (b) Calculate the sample means, variances, and covariances of b’X and c’X using (3-36). Compare the results in (a) and (b).

From (3-36):

Sample mean of **b’X** =

Sample mean of **c’X** =

Sample variance of **b’X** = **b’Sb**

Sample variance of **c’X** = **c’Sc**

Sample covariance of **b’X** and **c’X** = **b’Sc**

We will need to find the and S matrices before we find the sample matrices.

The matrix is:

## [,1]  
## [1,] 5  
## [2,] 3  
## [3,] 4

The S matrix is:

## [,1] [,2] [,3]  
## [1,] 13.0 -2.5 1.5  
## [2,] -2.5 1.0 -1.5  
## [3,] 1.5 -1.5 3.0

Now, we can calculate the sample summaries as according to (3-36).

## The sample mean of b'X is 12

## The sample mean of c'X is -1

## The sample variance of b'X is 12

## The sample variance of c'X is 43

## The sample covariance of b'X and c'X is -3

Let’s compare these results to the results from (a).

## The sample mean of bX is: 12

## The sample mean of cX is: 53

## The sample variance of bX is: 12

## The sample variance of cX is: 171

## The sample covariance of b'X and c'X is: 27

We see that the **b’X** sample mean and variance are the same in both (a) and (b). However, everything else is different. Everything in (a) is positive, while (b) has two negative values. There may have been some error in calculating in the sample summaries on my part in part (a), even if I was following the formulas provided in the book.

## 18. Energy consumption in 2001, by state, from the major sources

**= petroleum**

**= natural gas**

**= hydroelectric power**

**= nuclear electric power**

**is recorded in quadrillions in BTUs (Source: *Statistcal Abstract of the United States 2006*). The resulting mean and covariance matrix are**

### 

### (a) Using the summary statistics, determine the sample mean and variance of a state’s total energy consumption for these major sources.

Let be the total energy consumption.

Then, is:

## [,1]  
## [1,] 1.873

Then is:

## [,1]  
## [1,] 3.913

### 

### (b) Determine the sample mean and variance of the excess of petroleum consumption over natural gas consumption. Also find the sample covariance of this variable with the total variable in part a.

The excess of petroleum consumption over natural gas consumption will be: .

Then, is:

## [,1]  
## [1,] 0.258

Then, is:

## [,1]  
## [1,] 0.154

**Code Appendix**

knitr::opts\_chunk$set(echo = FALSE)  
library(ggplot2)  
library(ggExtra)  
library(ggpubr)  
library(matlib)  
library(plotly)  
library(plot3D)  
library(rgl)  
library(rglwidget)  
library(webshot)  
library(webshot2)  
Xmat <- c(3,4,6,-2,3,1)  
X <- matrix(Xmat, nrow = 3, ncol = 2, byrow = TRUE)  
one <- as.matrix(rep(1, dim(X)[1]))  
n <- dim(X)[1]  
sample\_mean <- 1/n\*t(X) %\*% one  
print(sample\_mean)  
plot(Xmat, pch = 19, col = 3, main = "Scatter plot in 2D", xlab = "X1", ylab = "X2")  
points(4,1, col = "purple1", pch = 19)  
sample\_mean  
mean <- matrix(c(4,4,4,1,1,1), ncol = 2)  
d <- X - mean  
colnames(d) <- c("d1","d2")  
vector <- rbind(5\*diag(3),t(d))  
rownames(vector) <- c("X", "Y", "Z", "d1","d2")  
#col <- c(rep("black",3))  
d  
d1 <- t(d[,1])  
d2 <- t(d[,2])  
rownames(d1) <- c("d1")  
rownames(d2) <- c("d2")  
# 3D  
vectors3d(d1, headlength = 0.05, ref.length = NULL, radius = 0.05,   
 labels = TRUE, cex.lab = 1.2, adj.lab = 0.5, frac.lab = 1.1)  
vectors3d(d2, headlength = 0.05, ref.length = NULL, radius = 0.05,   
 labels = TRUE, cex.lab = 1.2, adj.lab = 0.5, frac.lab = 1.1)  
x0 <- c(0, 0)  
y0 <- c(0, 0)  
z0 <- c(0, 0)  
x1 <- c(-1, 3)  
y1 <- c(2, -3)  
z1 <- c(-1, 0)  
cols <- c("mediumpurple1","maroon3")  
# 3D  
arrows3D(x0, y0, z0, x1, y1, z1, colvar = x1^2, col = cols,  
 lwd = 2, d = 3, main = "Part (b): Origin", bty ="b2", ticktype = "detailed")  
points3D(x0, y0, z0, add = TRUE, col="black",   
 colkey = FALSE, pch = 19, cex = 1)  
text3D(x1, y1, z1, c("d1", "d2"), colvar = x1^2, col = cols, add = TRUE, colkey = FALSE)  
sum1 <- rowSums(d1^2)   
sum2 <- rowSums(d2^2)  
l1 <- sqrt(sum1)  
l2 <- sqrt(sum2)  
# cat  
cat("The length of d1 is", l1, "\n")  
cat("The length of d2 is", l2)  
s11 <- (d1 %\*% t(d1))/3  
s22 <- (d2 %\*% t(d2))/3  
s12 <- (d1 %\*% t(d2))/3  
# cat  
cat("s11 is", s11, "\n")  
cat("s22 is", s22, "\n")  
cat("s12 is", s12, "\n")  
# matrix  
sMat <- c(s11, s12, s12, s22)  
S <- matrix(sMat, nrow = 2, ncol = 2, byrow = TRUE)  
S  
theta <- acos( sum(d1\*d2) / ( sqrt(sum(d1 \* d1)) \* sqrt(sum(d2 \* d2)) ) )  
angle <- cos(theta)  
angle  
rMat <- c(1, angle, angle, 1)  
R <- matrix(rMat, nrow = 2, ncol = 2, byrow = TRUE)  
R  
xMat <- c(-1, 3, -2,   
 2, 4, 2,   
 5, 2, 3)  
X <- matrix(xMat, nrow = 3, ncol = 3, byrow = TRUE)  
one <- as.matrix(rep(1, dim(X)[1]))  
n <- dim(X)[1]  
xbar <- 1/n\*t(X) %\*% one  
xbar  
dev <- X - one %\*% t(xbar)  
dev  
cat("d1 is [", dev[1, 1:3], "] \n")  
cat("d2 is [", dev[2, 1:3], "] \n")  
cat("d3 is [", dev[3, 1:3], "]")  
double <- dev %\*% t(dev)  
S <- double/2  
S  
det(S)  
S[1,1] + S[2,2] + S[3,3]  
dataMat <- c(3, 1, 0,   
 6, 4, 6,   
 4, 2, 2,   
 7, 0, 3,   
 5, 3, 4)  
data <- matrix(dataMat, nrow = 5, ncol = 3, byrow = TRUE)  
one <- as.matrix(rep(1, dim(data)[1]))  
n <- dim(data)[1]  
xbar <- 1/n\*t(data) %\*% one  
xbar  
mean <- matrix(data = 1, nrow = n) %\*% cbind(xbar[[1]], xbar[[2]], xbar[[3]])  
meanC <- data - mean  
meanC  
aMat <- c(1,1,-1)  
a <- matrix(aMat, nrow = 1, ncol = 3, byrow = TRUE)  
zero <- a %\*% t(meanC)  
a  
t(meanC)  
zero  
covar <- 1/(n-1) \* t(meanC) %\*% meanC  
covar  
a %\*% t(covar)  
data  
xMat <- c(1, 4, 3,  
 6, 2, 6,  
 8, 3, 3)  
X <- matrix(xMat, nrow = 3, ncol = 3, byrow = TRUE)  
bMat <- c(1,1,1)  
b <- matrix(bMat, nrow = 1, ncol = 3, byrow = TRUE)  
bX <- b %\*% X  
cMat <- c(1,2,-3)  
c <- matrix(cMat, nrow = 1, ncol = 3, byrow = TRUE)  
cX <- c %\*% X  
x1 <- t(b) \* X[1,1:3]  
bx1 <- b %\*% x1  
x2 <- t(b) \* X[2,1:3]  
bx2 <- b %\*% x2  
x3 <- t(b) \* X[3,1:3]  
bx3 <- b %\*% x3  
# cat  
cat("The observed value of bX1 is:", bx1, "\n")  
cat("The observed value of bX2 is:", bx2, "\n")  
cat("The observed value of bX3 is:", bx3)  
m1 <- (bx1 + bx2 + bx3)/3  
cat("The sample mean of bX is:", m1)  
v1 <- ((bx1 - m1)^2 + (bx2 - m1)^2 + (bx3 - m1)^2)/2  
cat("The sample variance of bX is:", v1)  
x1 <- t(c) \* X[1,1:3]  
cx1 <- c %\*% x1  
x2 <- t(c) \* X[2,1:3]  
cx2 <- c %\*% x2  
x3 <- t(c) \* X[3,1:3]  
cx3 <- c %\*% x3  
# mean/var  
m2 <- (cx1 + cx2 + cx3)/3  
v2 <- ((cx1 - m2)^2 + (cx2 - m2)^2 + (cx3 - m2)^2)/2  
# cat  
cat("The observed value of cX1 is:", cx1, "\n")  
cat("The observed value of cX2 is:", cx2, "\n")  
cat("The observed value of cX3 is:", cx3, "\n")  
cat("The sample mean of cX is:", m2, "\n")  
cat("The sample variance of cX is:", v2)  
num1 <- (bx1 - m1)\*(cx1 - m2)  
num2 <- (bx2 - m1)\*(cx2 - m2)  
num3 <- (bx3 - m1)\*(cx3 - m2)  
covar1 <- (num1 + num2 + num3)/2  
cat("The sample covariance of b'X and c'X is:", covar1)  
# xbar  
one <- as.matrix(rep(1, dim(X)[1]))  
n <- dim(X)[1]  
xbar <- 1/n\*t(X) %\*% one  
# S  
mean <- matrix(data = 1, nrow = n) %\*% cbind(xbar[[1]], xbar[[2]], xbar[[3]])  
meanX <- X - mean  
covar <- 1/(n-1) \* t(meanX) %\*% meanX  
xbar  
covar  
sm1 <- b %\*% xbar  
sm2 <- c %\*% xbar  
sV1 <- b %\*% covar %\*% t(b)  
sV2 <- c %\*% covar %\*% t(c)  
covar2 <- b %\*% covar %\*% t(c)  
# cat  
cat("The sample mean of b'X is", sm1, "\n")  
cat("The sample mean of c'X is", sm2, "\n")  
cat("The sample variance of b'X is", sV1, "\n")  
cat("The sample variance of c'X is", sV2, "\n")  
cat("The sample covariance of b'X and c'X is", covar2)  
cat("The sample mean of bX is:", m1, "\n")  
cat("The sample mean of cX is:", m2, "\n")  
cat("The sample variance of bX is:", v1, "\n")  
cat("The sample variance of cX is:", v2, "\n")  
cat("The sample covariance of b'X and c'X is:", covar1)  
xbarMat <- c(0.766, 0.508, 0.438, 0.161)  
xbar <- matrix(xbarMat, nrow = 4, ncol = 1, byrow = TRUE)  
sMat <- c(0.856, 0.635, 0.173, 0.096,   
 0.635, 0.568, 0.128, 0.067,   
 0.173, 0.127, 0.171, 0.039,  
 0.096, 0.067, 0.039, 0.043)  
S <- matrix(sMat, nrow = 4, ncol = 4, byrow = TRUE)  
oneMat <- c(1,1,1,1)  
one <- matrix(oneMat, nrow = 1, ncol = 4, byrow = TRUE)  
one %\*% xbar  
one %\*% S %\*% t(one)  
minMat <- c(1,-1,0,0)  
min <- matrix(minMat, nrow = 1, ncol = 4, byrow = TRUE)  
min %\*% xbar  
min %\*% S %\*% t(min)